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ON AMBIENT EMBEDDINGS IN THE HILBERT CUBE

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It is shown that for each closed subset X of codimension at least three in the Hilbert cube Q there is a Cantor set C in Q so that no homeomorphism of Q onto itself takes C into X.

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1. Introduction

Let M and N be subsets of a topological space X. We say M is *ambiently embeddable* in N if there is a homeomorphism h of X onto itself that carries M into N. We call a subset U of X an *ambiently universal* set for a family F if each set in F is ambiently embeddable in U.

Bothe has made an extensive study of ambiently universal sets in E^n [2, 3, 4, 5]. Recently the author [13] showed that each closed set X in E^n of dimension at most $n-3$ fails to be ambiently universal with respect to Cantor sets in E^n. From these works our understanding of ambiently universal sets in E^n is fairly complete.

The purpose of this note is to generalize the results of [13] to the Hilbert cube.

2. Definitions and notation

We let S^n, B^n, and E^n denote the n-sphere, the n-ball, and Euclidean n-space, respectively. We let I denote the interval $[-1, 1]$. The product of n copies of I will be denoted by I^n, and the Hilbert cube which is the product of countably infinite copies of I will be denoted by Q. A closed subset X of Q is said to have *codimension* $\geqslant k$ if $H_q(U, U-X)=0$ for $0\leqslant q<k$ and all open subsets U of Q, the

homology being taken with integer coefficients. We say X has *codimension* k if X has codimension $\geqslant k$ but does not have codimension $\geqslant k+1$ [9, 11].

3. Bing Cantor sets in Q

In E^n $(n \geqslant 3)$ wild Cantor sets may be described as the intersection of nested compact manifolds M_i so that each component of each M_i is homeomorphic to $B^2 \times S^{n-2}$ and contains exactly two components of M_{i+1} [1, 12].

Daverman has given a clever way of generalizing the Bing Cantor sets to the Hilbert cube [8]. Consider Bing's decomposition G of the 3-cube I^3 into points and tame arcs [1]. Bing showed that the decomposition space I^3/G is homeomorphic to an Alexander crumpled cube and then showed that two copies sewn together with the identity homeomorphism along the boundary yields a 3-sphere. Daverman considered the decomposition G' of $I^3 \times Q$ consisting of points and (arcs of G) $\times Q$. He showed that $(I^3 \times Q)/G'$ is homeomorphic to Q. The image of the nondegenerate elements of G' in Q forms a Cantor set that we also call a *Bing Cantor set*.

4. Ramified Bing Cantor sets in Q

By a 3-*ball with end disks* we mean a 3-ball B that has two disjoint disks D_1 and D_2 identified in the boundary of B. A 3-ball with end disks may be identified with $B^2 \times I$ in such a manner that the end disks are identified with $B^2 \times \{-1\}$ and $B^2 \times \{1\}$. Whenever a 3-ball is given in a factored form $D^2 \times [a, b]$, where D^2 is a 2-dimensional disk and $[a, b]$ is a closed interval, we consider $D^2 \times [a, b]$ to be a 3-ball with end disks $D^2 \times \{a\}$ and $D^2 \times \{b\}$.

Consider the 3-ball $B^2 \times I$. Consider n disjoint disks $D_1, D_2, \ldots, D_n$ in the interior of B^2. We call $\bigcup D_1 \times I$ an n-fold ramification of $B^2 \times I$ [8, 10, 13].

Consider the 3-balls T_1 and T_2 in $B^2 \times 1$ as shown in Fig. 1. The end disks of T_1 and T_2 are the intersections of T_1 and T_2 with the boundary of $B^2 \times I$. We call the embedding of $T_1 \cup T_2$ in $B^2 \times I$ a Bing embedding.

Let D^2 be a disk in the interior of I^2 and set $W_0 = D^2 \times I \subset I^3$. Inductively define W_{i+1} to be a finite collection of 3-balls with end disks by taking a Bing embedding in each component of W_i. The Bing decomposition of I^3 can be described as points and the components of $\bigcap W_i$. If care is taken the components of $\bigcap W_i$ will be arcs. The arcs simplify our description but are not essential to our proof.

Let $\alpha = a_0, a_1, a_2, \ldots$ be a sequence of positive integers. We construct a ramified Bing decomposition with respect to α of I^3 as points and the components of the intersection of nested manifolds $M_0, M_1, M_2, \ldots$. The set M_0 is, as before, $D^2 \times I$. Each component of each M_i is a 3-ball with end disks. Let i be a nonnegative integer. The manifold M_{2i+1} is obtained by taking an a_i-fold ramification in each

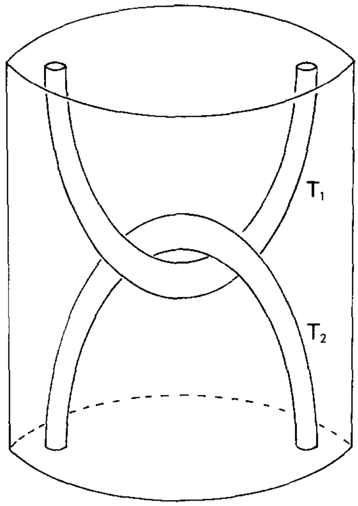

Fig. 1.

component of M_{2i}. The manifold M_{2i+2} is obtained by taking a Bing embedding in each component of M_{2i+1}. If care is taken each component of $\bigcap M_i$ will be an arc.

Let G be a ramified Bing decomposition of I^3. The Daverman argument also shows that the decomposition G' of $I^3 \times Q$ consisting of points and (arcs of $G) \times Q$ yields a space homeomorphic to Q. Once again the image of the nondegenerate elements is a wild Cantor set which we call a ramified Bing Cantor set.

5. Gropes and Bing decompositions

An *n-stage grope* is a 2-dimensional polyhedron A_n along with a decomposition into subpolyhedra $A_{-1} \subset A_0 \subset A_1 \subset A_2 \subset \cdots \subset A_n$ so that A_0 is a disk with handles whose boundary is A_{-1}. For $0 \leq i \leq n$, $A_i - A_{i-1}$ is a finite collection of open disks with handles. The closure of each component of $A_i - A_{i-1}$ is a closed disk with handles whose intersection with A_{i-1} is a handle curve of $A_{i-1} - A_{i-2}$ for $i > 0$, the correspondence between components of $A_i - A_{i-1}$ and handle curves of $A_{i-1} - A_{i-2}$ being one-to-one. See [6] for more details.

Theorem 5.1. *Let M_i be a defining sequence for a ramified Bing decomposition of I^3 corresponding to the sequence $\alpha = a_0, a_1, a_2, \ldots$. Let A_n be an n-stage grope with decomposition A_i. If f: $A_n \to I^3 - \bigcap M_i$ is a continuous function so that $f(A_{-1}) \subset I^3 - M_0$ and $f|A_{-1}$ is essential in $I^3 - M_0$, then $a_n \leq$ twice the number of handles of $A_n - A_{n-1}$.*

The proof of Theorem 5.1 is implicit in [13]; however, some comment is in order. The proofs in [13] avoid $n = 3$. These proofs deal with Bing decompositions of E^n where the nondegenerate elements are described by manifolds of the form $B^2 \times S^{n-2}$.

For $n>3$ the fundamental group of the boundary of $B^2\times S^{n-2}$ is isomorphic with the integers. For $n=3$ this is not the case. However, appropriate modifications do work for the Bing decomposition of I^3 instead of E^3.

Perhaps the easiest way to verify Theorem 5.1 is to take the 2-spin [7] of the ramified Bing decomposition of I^3. This gives a decomposition of S^5 that is of the type studied in [13] and the function f can be approximated by an embedding. Then Theorems 7.5, 7.8, and 8.2 of [13] may be invoked directly.

6. The main theorem

Theorem 6.1. *Let X be a closed subset of Q that has codimension $\geqslant 3$. There is a Cantor set C in Q so that C is not ambiently embeddable in X.*

Proof. Let $J_0, J_1, J_2, \ldots$ be a collection of simple closed curves in $Q-X$ that are dense in the space of embeddings of S^1 into Q. For each nonnegative integer n, let P_n be an n-stage grope in $Q-X$ that has a decomposition $A_{-1}\subset A_1\subset\cdots\subset A_n=P_n$ so that $A_{-1}=J_n$. The J_n and P_n exist because X has codimension $\geqslant 3$. Let b_n be the number of handles of A_n-A_{n-1} and set $a_n = 2b_n+1$.

Let M_i be a defining sequence for a ramified Bing decomposition of I^3 with respect to the sequence $a_0, a_1, a_2, \ldots$.

Let G be the resulting decomposition of I^3 and G' the decomposition of $I^3\times Q$ described by Daverman. We identify $(I^3\times Q)/G'$ with Q. The image of the nondegenerate elements of G' is a ramified Bing Cantor set C in Q. Suppose C can be pushed into X by a homeomorphism of Q. Then we may assume that the identification of $(I^3\times Q)/G'$ with Q takes the image of the nondegenerate elements of G' into X. Hence, with this identification, we may consider the sets J_i and P_i to be contained in $I^3\times Q$ and missing the nondegenerate elements of G'. Since the simple closed curves J_i are dense, for some n the projection π: $I^3\times Q\to I^3$ restricted to P_n gives a map of a grope into $I^3-\bigcap M_i$ so that $\pi(J_n)\cap M_0$ is empty and $\pi|J_n$ is essential in I^3-M_0. Theorem 5.1 implies that $a_n\leqslant 2b_n$. But $a_n=2b_n+1$, a contradiction. □

It is known [14] that a closed set in Q that is not strongly infinite dimensional must have infinite codimension. Hence we get the following.

Corollary 6.2. *Let X be a closed subset of Q that is not strongly infinite dimensional (this includes finite dimensional and countably infinite dimensional sets). Then there is a ramified Bing Cantor set C so that C is not ambiently embeddable in X.*

References

[1] R.H. Bing, A homeomorphism between the 3-sphere and the sum of two solid horned spheres, Ann. of Math. 56 (2) (1952) 354–362.

[2] H.G. Bothe, Universalmengen bezuglich der Lage im E^n, Fund. Math. 56 (1964) 203-212.
[3] H.G. Bothe, Zur Lage null- und eindimensionaler Punktmengen in E^3, Fund. Math. 58 (1966) 1-30.
[4] H.G. Bothe, Zwei Universalmengen in euklidischen Raum, Math. Nachr. 39 (1969) 117-133.
[5] H.G. Bothe, Ein eindimensionales Kompaktum in E^3, das sich nicht lagetreu in die Mengersche Universalkurve einbetten laesst, Fund. Math. 54 (1964) 251-258.
[6] J.W. Cannon, The recognition problem: What is a topological manifold?, Bull. Amer. Math. Soc. 84 (1978) 832-866.
[7] J.W. Cannon and R.J. Daverman, Cell-like decompositions arising from mismatched sewings: applications to 4-manifolds, Fund. Math. 111 (1981) 211-233.
[8] R.J. Daverman, Infinite inflations of crumpled cubes, Topology and its Applications 12 (1981) 35-43.
[9] R.J. Daverman and J.J. Walsh, Čech homology characterizations of infinite dimensional manifolds, Amer. J. of Math. 103 (1981) 411-435.
[10] W.T. Eaton, A generalization of the dog bone space to E^n, Proc. Amer. Math. Soc. 39 (1973) 379-387.
[11] D.J. Garity, Aleksandrov's problem, Pol's example and an alternate approach to dimension, Proceedings of the BYU Topology Conference 1984, to appear.
[12] L.L. Lininger, Actions on S^n, Topology 9 (1970) 301-308.
[13] D.G. Wright, Ambiently universal sets in E^n, Trans. Am. Math. Soc. 277 (1983) 655-664.
[14] D.G. Wright, On sets with finite codimension in the Hilbert cube, Proceedings of the BYU Topology conference 1984, 17-23.